# October Math Gems

PROBLEM OF THE WEEK 18

## §1 problems

**Problem 1.1.** Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the two other angles.

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**Problem 1.2.** Prove that the only solution of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

in integers $x, y$, and $z$ is $x = y = z = 0$.

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**Problem 1.3.** Show that each number of the sequence

$$49, 4489, 444889, 44448889, \ldots$$

is a perfect square.(Recall the formula for the sum of a geometric progression)

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**Problem 1.4.** Solve the system

$$\begin{aligned} 2x^2 - 4xy + 3y^2 &= 36 \\ 3x^2 - 4xy + 2y^2 &= 36 \end{aligned}$$

(One solution is easy to guess, but you are required to find all solutions. Knowledge of analytic geometry is not needed to solve this problem, but may help to understand the result-how?)

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**Problem 1.5.** Solve the following system of three equations for the unknowns $x, y$, and $z$ ( $a, b$, and $c$ are given ) :

$$\begin{aligned} x^2y^2 + x^2z^2 &= axyz, \\ y^2z^2 + y^2x^2 &= bxyz, \\ z^2x^2 + z^2y^2 &= cxyz. \end{aligned}$$

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**Problem 1.6.** Prove: If $n$ is an integer greater than $1, n^{n-1} - 1$ is divisible by $(n-1)^2$.

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**Problem 1.7.** Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbors receive jointly. Is there just one way to distribute the money? Prove your answer.

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**Problem 1.8.** If the coefficients of $x^{-2}$ and $x^{-4}$ in the expansion of

$$\left(x^{\frac{1}{3}} + \frac{1}{2x^{\frac{1}{3}}}\right) \quad (x > 0)$$

are $m$ and $n$ respectively, then compute $\frac{m}{n}$.

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**Problem 1.9.** The coefficients of $x^{50}$ in the expansion of

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$$

is [1002]50. Prove or Disprove.

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**Problem 1.10.** Suppose $x$ and $y$ are nonzero real numbers simultaneously satisfying the following system of equations

$$x + \frac{2018}{y} = 1000$$

$$\frac{9}{x} + y = 1$$

Find the maximum possible value of $x + 1000y$.

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**Problem 1.11.** From the following system of equations

$$x^2 - y^2 = 9$$

$$xy = 3$$

The value of of $x + y$ can be written in the form of $\pm\sqrt{\sqrt{a} + b}$, then find the values of $a$ and $b$.

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**Problem 1.12.** Let $a$ and $b$ be distinct real numbers. Solve the following equation

$$\sqrt{x - b^2} - \sqrt{x - a^2} = a - b$$

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**Problem 1.13.**

$$\log_2(11 - 6x) = 2\log_2(x - 1) + 3$$

find the value x.

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**Problem 1.14.** Solve

$$cos^2(\frac{x}{2}) = cos^2(x)$$

, on the interval $0 \leq x < 2\pi$.

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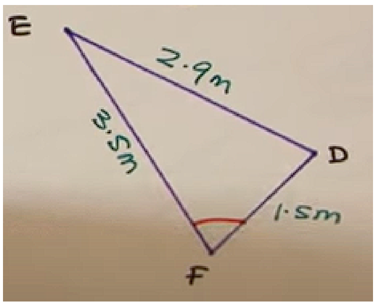
**Problem 1.15.** Find angle F



**Problem 1.16.** Determine the range of the function

$$f(x) = 4 - 2sin(3x)$$

**Problem 1.17.** The surface area of a human's lungs is equal to half of a tennis court (2100 $ft^2$). How many square feet would the lungs of a 30-person baseball team cover?

**Problem 1.18.** At a shop in Times Square one "I LOVE NY" t-shirt is sold every 10 minutes for 19.95 dollars each. The shop opens from 9 am until 9 pm every day.
How many t-shirts are sold in a week?

**Problem 1.19.**

$$(x + 3)(x - 5) = 5$$

Find all the solutions to the equation above.

**Problem 1.20.** Find the area of the blue part, where $r = 4$.